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# Exponential generating functions for the associated Bessel functions

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**Abstract**
Similar to the associated Legendre functions, the differential equation for the associated Bessel functions $B_{l,m}(x)$ is introduced so that its form remains invariant under the transformation $l \to -l-1$. A Rodrigues formula for the associated Bessel functions as squared integrable solutions in both regions $l < 0$ and $l \geqslant 0$ is presented. The functions with the same $m$ but with different positive and negative values of $l$ are not independent of each other, while the functions with the same $l+m$ $(l-m)$ but with different values of $l$ and $m$ are independent of each other. So, all the functions $B_{l,m}(x)$ may be taken into account as the union of the increasing (decreasing) infinite sequences with respect to $l$. It is shown that two new different types of exponential generating functions are attributed to the associated Bessel functions corresponding to these rearranged sequences.

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## 1. Introduction and motivation

The generating functions have found many applications in the physical, chemical and mathematical systems. The study of the important chance process called the branching process [1–3], random graphs and complex networks [4], polymerization kinetics [5], counting problems in combinatorics [6], are some applications of the theory of generating functions. The generating functions are used to obtain expected values (averages), variances, moments and cumulants of distributions, and also, to establish relationships between distributions [7].

The exponential generating functions and their numerous generalizations have been alternatively introduced and studied by various methods for the orthogonal polynomials and special functions (see [8–22]). The generating functions as continuous functions generally describe the convergence of an infinite summation of given infinite sequences of functions.

In this sense, the special polynomials and functions of a given sequence in one variable $x$ can be defined as the coefficients in the expansion of their generating functions. The generating functions include various useful properties and all the information that is needed to generate the solutions corresponding to a differential equation or a set of recursion relations between those solutions. Therefore, generating functions are very useful to analyze problems involving summations on the infinite sequences of functions such as coherent states. The application of generating functions for known orthonormal special functions allows one to derive a compact formula for the coherent states. Generating function corresponding to a given set of special functions is not unique. This paper has been devoted to introducing new generating functions for the solutions of the differential equation of associated Bessel functions which can be applied to obtain bound states of some solvable models in the framework of supersymmetric quantum mechanics, such as radial bound states of the hydrogen-like atoms [23]. Then, let us remember that Krall and Frink have first studied the Bessel polynomials in the formalism of hypergeometric functions [24]. Also, some authors have introduced some generating functions for Bessel polynomials [25–27]. Moreover, the generating functions associated with the group-theoretic techniques and the Stirling numbers of the second kind have been derived for the special sequences of the generalized Bessel polynomials [28–30]. Two different types of $q$ analogues of the generating functions for generalized Bessel polynomials have been calculated in [31] too.

In comparison with the Bessel and Romanovski differential equations [32], the Hermite, Laguerre and Jacobi ones from the viewpoint of their polynomial solutions application to the physics problems, have attracted much attention until now. However, the Bessel and Romanovski polynomials have also been applied to obtain the wavefunctions of some of the physical potentials. For example, we mention the factorization methods for the differential equation of associated Bessel polynomials which are applied to obtain the supersymmetric structures corresponding to the radial bound states of the hydrogen atom [23]. The solutions of the Schrödinger equation for some noncentral potentials such as hyperbolic Scarf and trigonometric Rosen–Morse [32–34], and of the Klein–Gordon equation with scalar and vector potentials, are obtained in terms of the Romanovski polynomials [35]. The trigonometric Rosen–Morse potential is an appropriate candidate to describe the quark–gluon dynamics in QCD, since, it can be considered as an appropriate approximation of the combination of the Coulomb, the infinite wall and the linear potentials. Therefore, the Bessel and Romanovski nonclassical polynomials have the merit of taking more into account in order to derive new relations.

The paper has two sections. In section 2, we introduce the differential equation of associated Bessel functions $B_{l,m}^{(q,\beta)}(x)$ in terms of the indices $l$ and $m$, in similarity with associated Legendre functions $P_{l,m}(x)$:

$$(1-x^2)P''_{l,m}(x) - 2xP'_{l,m}(x) + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_{l,m}(x) = 0.$$

Despite of the $P_{l,m}(x)$'s, the differential equation for the associated Bessel functions is altered when $m$ is replaced by $-m$. In conclusion, the function $B_{l,-m}^{(q,\beta)}(x)$ is not another solution for it. However, the differential equation is invariant under transformation $l \to -l-1$, and it, in turn, leads to considering the solutions as $B_{l,m}^{(q,\beta)}(x)$ with $l < 0$, too. Consequently, we can offer a Rodrigues formula for $B_{l,m}^{(q,\beta)}(x)$ as the squared integrable solutions with $l < 0$ and $l \geqslant 0$. Furthermore, simultaneous realization of laddering equations with respect to $l$ and $m$ by the given Rodrigues formula in both regions, is considered. In section 3, it is shown that the independent solutions $B_{l,m}^{(q,\beta)}(x)$ are classified in three different types of infinite sequences. The first-type sequences of the associated Bessel functions have the same $l$ but different $m$.

The generating functions corresponding to them can be followed via the known generating functions of the Bessel polynomials. The second and the third types of the sequences are constituted by the independent associated Bessel functions with the same $l+m$ and $l-m$, respectively. These two later types provide the possibility to view the set of the independent associated Bessel functions in two new perspectives, different from the first one. Finally, we calculate two new kinds of generating functions for any of these types of sequences depending on whether $l+m$ and $l-m$ are odd or even.

## 2. Associated Bessel functions

This section includes, in addition to review some results of [36], the formulation of Rodrigues representation for the associated Bessel functions $B_{l,m}^{(q,\beta)}(x)$ with $l<0$. Let us first recall that the generalized Bessel polynomials of degree $n$ [37], i.e.

$$B_n^{(\alpha,\beta)}(x) = \frac{a_n(\alpha,\beta)}{x^\alpha\,\mathrm{e}^{-\frac{\beta}{x}}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(x^{\alpha+2n}\,\mathrm{e}^{\frac{-\beta}{x}}\right) = a_n(\alpha,\beta)\beta^n y_n(x;\alpha+2,\beta), \tag{1}$$

are eigenfunctions of the following linear second order differential operator:

$$x^{-\alpha}\,\mathrm{e}^{\frac{\beta}{x}}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\alpha+2}\,\mathrm{e}^{\frac{-\beta}{x}}\frac{\mathrm{d}}{\mathrm{d}x}\right)B_n^{(\alpha,\beta)}(x) = n(n+\alpha+1)B_n^{(\alpha,\beta)}(x), \tag{2}$$

where $a_n(\alpha,\beta)$'s are the normalization coefficients. It must be pointed out that the $y_n(x;\alpha,\beta)$ representation of the generalized Bessel polynomials is given by [29, 30]

$$y_n(x;\alpha,\beta) = \sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+n+k-2}{k}k!\left(\frac{x}{\beta}\right)^k = \left(\frac{-x}{\beta}\right)^n n!L_n^{(1-\alpha-2n)}\left(\frac{\beta}{x}\right). \tag{3}$$

For $\beta>0$ and $\alpha<-2$, the generalized Bessel polynomials are orthogonal and square integrable with respect to the weight function $x^\alpha\,\mathrm{e}^{\frac{-\beta}{x}}$ in the interval $0\leqslant x<\infty$. Choosing $n=l-m+\frac{q}{2}$ and $\alpha=2m-q$ in the differential equation (2), it is straightforward to show that the associated Bessel functions [36]

$$B_{l,m}^{(q,\beta)}(x) := \frac{a_{l,m}(q,\beta)}{a_{l-m+\frac{q}{2}}(2m-q,\beta)}x^m B_{l-m+\frac{q}{2}}^{(2m-q,\beta)}(x) = \frac{a_{l,m}(q,\beta)}{x^{m-q}\,\mathrm{e}^{\frac{-\beta}{x}}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l-m+\frac{q}{2}}\left(x^{2l}\,\mathrm{e}^{\frac{-\beta}{x}}\right), \tag{4}$$

with $l-m+\frac{q}{2}\geqslant 0$ as a non-negative integer, satisfy the following differential equation:

$$x^2 B_{l,m}^{\prime\prime(q,\beta)} + [(2-q)x+\beta]\,B_{l,m}^{\prime(q,\beta)} - \left[\left(l+\frac{q}{2}\right)\left(l-\frac{q}{2}+1\right)+\frac{m\beta}{x}\right]B_{l,m}^{(q,\beta)} = 0. \tag{5}$$

Similar to the associated Legendre differential equation, since equation (5) is unaltered when $l$ is replaced by $-l-1$, the functions $B_{l,m}^{(q,\beta)}(x)$ with negative $l$ are also another solutions for it. But contrary to the associated Legendre differential equation, which is quadratic in terms of $m$, the associated Bessel differential equation (5) is linear in terms of it. It will be clear from our discussions that the functions $B_{l,m}^{(q,\beta)}$ with $m\leqslant l+\frac{q}{2}$ and $m\geqslant -l+\frac{q}{2}$ are not normalized by the weight function $x^{-q}\,\mathrm{e}^{\frac{-\beta}{x}}$. Therefore, we limit our study to $m-\frac{q}{2}\leqslant l\leqslant\frac{q}{2}-m-1$ with $m\leqslant\frac{q-1}{2}$, in which $q$ is an integer number. In figure 1, we have schematically shown the two-fold hierarchy of all the associated Bessel functions corresponding to $q=6$ as the points $(l,m)$ on a flat plane whose horizontal and vertical axes are labeled with $l$ and $m$, respectively. Note that for an odd integer $q$, one of the two parameters $l$ and $m$ has to be half-integer, and the other has to be integer. The Rodrigues formula (4) shows that the associated Bessel functions $B_{l,m}^{(q,\beta)}$ are finite summations of (not necessarily positive) integer and half-integer powers of $x$ when $q$ is even and odd, respectively.

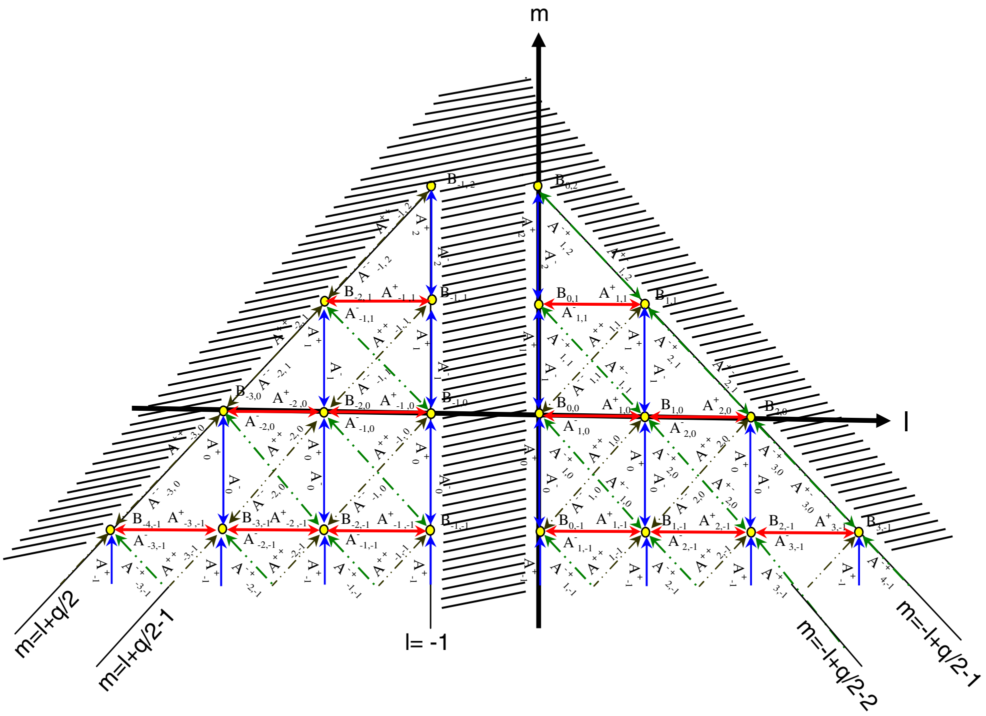


**Figure 1.** The comprehensive plan of the squared-integrable solutions for the differential equation (3) of associated Bessel functions with $q = 6$.

Now, in order to formulate the laddering relations with respect to $l$ and $m$, and also to realize the square integrability condition, it is necessary that we obtain the highest powers of $x$ in the associated Bessel functions $B_{l,m}^{(q,\beta)}(x)$ for $l < 0$ and $l \geqslant 0$, respectively:

$$B_{l,m}^{(q,\beta)}(x) = \begin{cases} a_{l,m}(q,\beta)(-1)^{m-l-\frac{q}{2}} \frac{\Gamma\left(-l-m+\frac{q}{2}\right)}{\Gamma(-2l)} x^{l+\frac{q}{2}} + O(x^{l+\frac{q}{2}-1}) & l < 0 \quad (6a) \\ a_{l,m}(q,\beta)(-1)^{-m-l+\frac{q}{2}-1} \beta^{2l+1} \frac{\Gamma\left(l-m+\frac{q}{2}+1\right)}{\Gamma(2l+2)} x^{-l+\frac{q}{2}-1} + O(x^{-l+\frac{q}{2}-2}) & l \geqslant 0. \quad (6b) \end{cases}$$

Note that (6$a$) is directly derived using (4), while (6$b$) is followed from the following relation and also (6$a$),

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2l+1} \left(x^{2l}\,\mathrm{e}^{-\frac{\beta}{x}}\right) = \beta^{2l+1} x^{-2l-2}\,\mathrm{e}^{-\frac{\beta}{x}} \qquad l \geqslant 0. \tag{7}$$

The inductive reasoning can be applied to prove relation (7). The orthogonality of the associated Bessel functions for a given $m$, and also their square integrability for $l < 0$ and $l \geqslant 0$ are obtained, respectively, as

$$\int_0^\infty B_{l,m}^{(q,\beta)} B_{l',m}^{(q,\beta)} x^{-q}\,\mathrm{e}^{\frac{-\beta}{x}}\,\mathrm{d}x = \delta_{ll'} a_{l,m}^2(q,\beta) \frac{\Gamma\left(-l-m+\frac{q}{2}\right)\Gamma\left(l-m+\frac{q}{2}+1\right)}{\beta^{-2l-1}(\mp 2l \mp 1)}. \tag{8}$$

They follow from relations (4), (6) and applying integration by parts $l-m+\frac{q}{2}$ and $-l-m+\frac{q}{2}-1$ times, respectively. In the $k$th stage of these processes, the total differential expressions become zero, because $l'+m-\frac{q}{2}+k+1 < 0$ and $-l'+m-\frac{q}{2}+k < 0$.

Similar to [36], the associated Bessel differential equation (5) can be simultaneously factorized by the ladder operators

$$\begin{aligned} A^{\pm}_{l,m} &= \pm x^2\frac{\mathrm{d}}{\mathrm{d}x} + \left(l \mp \frac{q}{2}\right)x \pm \frac{\left(l \pm m \mp \frac{q}{2}\right)\beta}{2l}, \\ A^{\pm}_{m} &= \pm x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\beta}{2x} \pm \frac{\beta}{2x} - m + \frac{1}{2}(1+q) \pm \frac{1}{2}(1-q), \end{aligned} \tag{9}$$

with the following eigenvalues:

$$E_{l,m} = \frac{\left(l-m+\frac{q}{2}\right)\left(-l-m+\frac{q}{2}\right)\beta^2}{4l^2}, \qquad \mathcal{E}_{l,m} = \left(\frac{q}{2}-l-m\right)\left(l-m+\frac{q}{2}+1\right), \tag{10}$$

as shape invariance symmetry equations for the indices $(l, m)$ and $(l-1, m)$ as well as $(l, m)$ and $(l, m-1)$, respectively:

$$A^{+}_{l,m}A^{-}_{l,m}B^{(q,\beta)}_{l,m}(x) = E_{l,m}B^{(q,\beta)}_{l,m}(x) \qquad A^{-}_{l,m}A^{+}_{l,m}B^{(q,\beta)}_{l-1,m}(x) = E_{l,m}B^{(q,\beta)}_{l-1,m}(x), \tag{11a}$$

$$A^{+}_{m}A^{-}_{m}B^{(q,\beta)}_{l,m}(x) = \mathcal{E}_{l,m}B^{(q,\beta)}_{l,m}(x) \qquad A^{-}_{m}A^{+}_{m}B^{(q,\beta)}_{l,m-1}(x) = \mathcal{E}_{l,m}B^{(q,\beta)}_{l,m-1}(x). \tag{11b}$$

In this paper, we are interested to present the raising and lowering relations of the indices $l$ and $m$ of the associated Bessel functions for both regions, $l<0$ and $l \geqslant 0$ of $m-\frac{q}{2} \leqslant l \leqslant \frac{q}{2}-m-1$. The shape invariance equations with respect to $l$ and $m$ are realized for every normalization coefficient. However, realization of the laddering equations

$$A^{+}_{l,m}B^{(q,\beta)}_{l-1,m}(x) = \sqrt{E_{l,m}}B^{(q,\beta)}_{l,m}(x) \qquad A^{-}_{l,m}B^{(q,\beta)}_{l,m}(x) = \sqrt{E_{l,m}}B^{(q,\beta)}_{l-1,m}(x), \tag{12a}$$

$$A^{+}_{m}B^{(q,\beta)}_{l,m-1}(x) = \sqrt{\mathcal{E}_{l,m}}B^{(q,\beta)}_{l,m}(x) \qquad A^{-}_{m}B^{(q,\beta)}_{l,m}(x) = \sqrt{\mathcal{E}_{l,m}}B^{(q,\beta)}_{l,m-1}(x), \tag{12b}$$

imposes two recursion relations with respect to $l$ and $m$, respectively, on the coefficients $a_{l,m}(q,\beta)$. One can easily show that the laddering equations (12$a$) and (12$b$) as well as their corresponding recursion relations for the normalization coefficients $a_{l,m}(q,\beta)$, are simultaneously established if the latter are chosen as

$$a_{l,m}(q,\beta) = \begin{cases} \dfrac{\beta^{-l}(-1)^{\frac{q}{2}-m}}{\sqrt{\Gamma\left(l-m+\frac{q}{2}+1\right)\Gamma\left(-l-m+\frac{q}{2}\right)}} & m-\frac{q}{2} \leqslant l < 0 \qquad (13a) \\ \dfrac{\beta^{-l-1}(-1)^{-l-m+\frac{q}{2}-1}}{\sqrt{\Gamma\left(l-m+\frac{q}{2}+1\right)\Gamma\left(-l-m+\frac{q}{2}\right)}} & 0 \leqslant l \leqslant \frac{q}{2}-m-1, \qquad (13b) \end{cases}$$

where $m \leqslant \frac{q-1}{2}$. One cannot actually follow the raising and lowering forms from shape invariance symmetries by means of the arbitrary normalization coefficients. The point is in order to realize the laddering symmetries (12$a$) and (12$b$), it is a necessary condition to select the normalization coefficients as in (13$a$) and (13$b$).

Our reason for making the associated Bessel functions with $l<0$ goes back to the fact that the functions $B^{(q,\beta)}_{0,m}(x)$ with $m \leqslant \frac{q-1}{2}$ are not annihilated by the operators $A^{-}_{0,m}$. Therefore, for a given $m$, decreasing of the index $l$ can be terminated at $l = m-\frac{q}{2}$. Indeed, from equations (12$a$) we have $A^{+}_{\frac{q}{2}-m,m}B^{(q,\beta)}_{\frac{q}{2}-m-1,m}(x) = 0$ and $A^{-}_{m-\frac{q}{2},m}B^{(q,\beta)}_{m-\frac{q}{2},m}(x) = 0$. Moreover, from equations (12$b$), it becomes clear that the associated Bessel functions lain on the lines $l = m-\frac{q}{2}$ and $l = -m+\frac{q}{2}-1$ are annihilated by the ladder operators shifting $m$: $A^{+}_{l+\frac{q}{2}+1}B^{(q,\beta)}_{l,l+\frac{q}{2}}(x) = 0$ and $A^{+}_{\frac{q}{2}-l}B^{(q,\beta)}_{l,\frac{q}{2}-l-1}(x) = 0$. According to the above discussions, for a given non-negative $l$ and for every $m \leqslant \frac{q-1}{2}$, we have $B^{(q,\beta)}_{-l-1,m}(x) = \beta(-1)^{l+1} B^{(q,\beta)}_{l,m}(x)$, which means the functions with $l < 0$ and $l \geqslant 0$ settled on the horizontal lines of figure 1 are mutually dependent on each other. However, each of the vertical and oblique lines possess the associated Bessel functions that are independent of each other. This allows us to construct the generating functions for them in three different methods by using Rodrigues formula (4) for both regions $l < 0$ and $l \geqslant 0$. Note that the differential equation (5) has two independent solutions and above discussions have focused on one of them.

## 3. Exponential generating functions for the associated Bessel functions

The square-integrable associated Bessel functions can be applied to obtain bound states corresponding to some one-dimensional supersymmetric potentials and also some two-dimensional quantum-mechanical models having a Lie algebra symmetry [23, 38]. Therefore, exponential generating functions corresponding to the formal power series of associate Bessel functions $B^{(q,\beta)}_{l,m}(x)$ with the same $l$, the same $l+m$ and the same $l-m$, are important not only from the point of view of mathematical derivation but also from the point of view of physical applications. What we do is to consider three different methods for computing the generating functions, based on the presentation of appropriate infinite sequences of the associated Bessel functions. The first type of the infinite sequences is $\left\{B^{(q,\beta)}_{l,m}(x)\right\}^{-\infty}_{m=\frac{q}{2}+l}$ for $l < 0$ or $\left\{B^{(q,\beta)}_{l,m}(x)\right\}^{-\infty}_{m=\frac{q}{2}-l-1}$ for $l \geqslant 0$. Using the definitions $n := \frac{q}{2} - l - m - 1$ and $p := l - m + \frac{q}{2}$, all the associated Bessel functions $B^{(q,\beta)}_{l,m}(x)$ with $l < 0$ and $l \geqslant 0$ can also be rearranged as the union of the second- and third-type infinite sequences: $\left\{B^{(q,\beta)}_{l,-l-n+\frac{q}{2}-1}(x)\right\}^{\infty}_{n=0}$ and $\left\{B^{(q,\beta)}_{l,l-p+\frac{q}{2}}(x)\right\}^{\infty}_{p=0}$, respectively. The second-type sequences are increasing with respect to the index $l$ of the associated Bessel functions $B^{(q,\beta)}_{l,m}(x)$ while the third-type sequences are decreasing. These sequences are automatically covered by the associated Bessel functions which are linearly independent on the interval $0 \leqslant x < \infty$ with respect to the inner product (8), as $\left\{B^{(q,\beta)}_{\frac{q}{2}-m-2k-2,m}(x)\right\}^{-\infty}_{m=\frac{q}{2}-k-1} \bigcup \left\{B^{(q,\beta)}_{\frac{q}{2}-m-2k-1,m}(x)\right\}^{-\infty}_{m=\frac{q}{2}-k-1}$ and $\left\{B^{(q,\beta)}_{m+2k-\frac{q}{2}+1,m}(x)\right\}^{-\infty}_{m=\frac{q}{2}-k-1} \bigcup \left\{B^{(q,\beta)}_{m+2k-\frac{q}{2},m}(x)\right\}^{-\infty}_{m=\frac{q}{2}-k-1}$ with $k = 0, 1, 2, \ldots$. Now we are in a position that for the sequences given in above we can derive three different types of the generating functions.

### 3.1. Generating functions for given $q$ and $l$

Let us first introduce the generating functions for the power series with different $m$ but with the same $l$, which are calculated in a way similar to those for generalized Bessel polynomials [28–30]. The generating functions corresponding to the first-type sequences of associated Bessel functions with the different $m$ but the same $l$,

$$G_l(x,t) := \begin{cases} \displaystyle\sum_{m=0}^{\infty} \frac{t^m}{m!} \frac{B^{(q,\beta)}_{l,l-m+\frac{q}{2}}(x)}{a_{l,l-m+\frac{q}{2}}(q,\beta)} & \text{for} \quad l < 0 \qquad (14a) \\ \displaystyle\sum_{m=1}^{\infty} \frac{t^m}{m!} \frac{B^{(q,\beta)}_{l,l-m+\frac{q}{2}}(x)}{a_{l,l-m+\frac{q}{2}}(q,\beta)} & \text{for} \quad l \geqslant 0, \qquad (14b) \end{cases}$$

for $|t| < 1$, are

$$G_l(x,t) = (1+t)^{2l} x^{l+\frac{q}{2}} \, \mathrm{e}^{\frac{\beta t}{x(1+t)}}. \qquad (15)$$

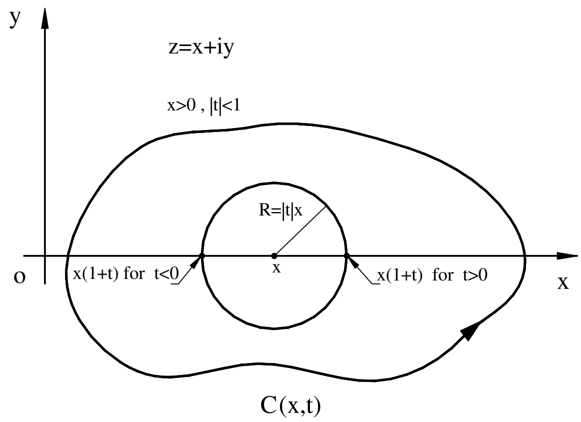


**Figure 2.** Plot of the integration contour $\mathcal{C}(x,t)$ for the generating functions of first-type sequences of the associated Bessel functions.

Relation (15), for both regions $l < 0$ and $l \geqslant 0$, are followed by the following Cauchy's integral formula:

$$\frac{1}{\Gamma(k+1)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k}\left(x^{2l}\,\mathrm{e}^{\frac{-\beta}{x}}\right) = \frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}(x,t)}\mathrm{d}z\frac{z^{2l}\,\mathrm{e}^{\frac{-\beta}{z}}}{(z-x)^{k+1}}, \tag{16}$$

with $k = l - m + \frac{q}{2}$. Figure 2 shows that the integration contour $\mathcal{C}(x,t)$ is a closed path around the circle $|z-x| = |t|\,x$ in the positive direction. $z = 0$ has also been settled out of the contour $\mathcal{C}(x,t)$. Also, the real pole $z = x(1+t)$ is always settled inside the contour, since it is on the circle at the right- and left-hand sides of center whether $t$ is positive or negative. Considering the following relation:

$$B_{l,m}^{(q,\beta)}(x) = a_{l,m}(q,\beta)(-1)^{l-m+\frac{q}{2}}x^{l+\frac{q}{2}}\,\Gamma\left(l-m+\frac{q}{2}+1\right)L_{l-m+\frac{q}{2}}^{-2l-1}\left(\frac{\beta}{x}\right), \tag{17}$$

one can establish the connection between the $G_l(x,t)$ with the Laguerre generating function (1.18) of [30].

### 3.2. *Generating functions for given q and l + m*

In this case the sequences are increasing with respect to $l$. Due to the fact that whether $n$ is odd or even, i.e. $n = 2k+1$ or $n = 2k$, the highest functions are $B_{-k-1,\frac{q}{2}-k-1}^{(q,\beta)}(x)$ and $B_{-k,\frac{q}{2}-k-1}^{(q,\beta)}(x)$, respectively. These functions lie on the lines $m = l + \frac{q}{2}$ and $m = l + \frac{q}{2} - 1$ of figure 1, respectively. Therefore, it is obvious that the terminology of highest functions has been devoted to the associated Bessel functions $B_{l,m}^{(q,\beta)}(x)$ with the most value for $m$. (a) First we suppose that $n$ is odd, i.e. $n = 2k+1$. For a given value of $k$, the generating functions corresponding to the second-type sequences are calculated as

$$\begin{aligned} G_{n=2k+1}(x,t) &= \sum_{m=0}^{\infty}\frac{t^m}{(2m)!}\frac{B_{m-k-1,\frac{q}{2}-m-k-1}^{(q,\beta)}(x)}{a_{m-k-1,\frac{q}{2}-m-k-1}(q,\beta)} \\ &= x^{\frac{q}{2}+k+1}\,\mathrm{e}^{\frac{\beta}{x}}\sum_{m=0}^{\infty}\frac{(xt)^m}{(2m)!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2m}\left(x^{2m-2k-2}\,\mathrm{e}^{\frac{-\beta}{x}}\right) \end{aligned}$$

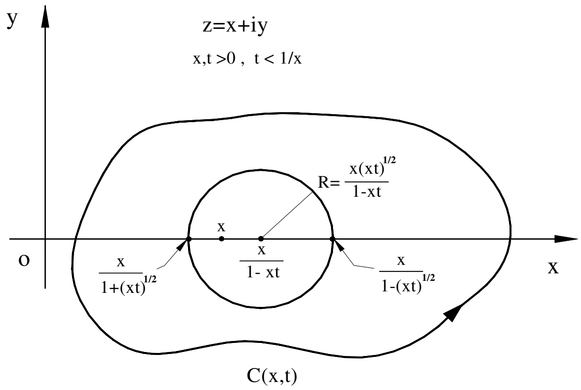


**Figure 3.** Plot of the integration contour $C(x,t)$ for the generating functions of second-type sequences of the associated Bessel functions.

$$
\begin{aligned}
&= x^{\frac{q}{2}+k+1}\,\mathrm{e}^{\frac{\beta}{x}} \sum_{m=0}^{\infty} \frac{(xt)^m}{2\pi\mathrm{i}} \oint_{C(x,t)} \mathrm{d}z \frac{z^{2m-2k-2}\,\mathrm{e}^{\frac{-\beta}{z}}}{(z-x)^{2m+1}} \\
&= \frac{x^{\frac{q}{2}+k+1}\,\mathrm{e}^{\frac{\beta}{x}}}{4\pi\mathrm{i}x\sqrt{xt}} \left[ \oint_{C(x,t)} \mathrm{d}z \frac{z^{-2k-2}(z-x)\,\mathrm{e}^{\frac{-\beta}{z}}}{z-\frac{x}{1-\sqrt{xt}}} - \oint_{C(x,t)} \mathrm{d}z \frac{z^{-2k-2}(z-x)\,\mathrm{e}^{\frac{-\beta}{z}}}{z-\frac{x}{1+\sqrt{xt}}} \right] \\
&= \frac{1}{2} x^{\frac{q}{2}-k-1} \left[ (1-\sqrt{xt})^{2k+1}\,\mathrm{e}^{\beta\sqrt{\frac{t}{x}}} + (1+\sqrt{xt})^{2k+1}\,\mathrm{e}^{-\beta\sqrt{\frac{t}{x}}} \right]. \qquad (18)
\end{aligned}
$$

As has been shown in figure 3, $C(x,t)$ is a closed contour in the positive direction around a circle with center at $\left(\frac{x}{1-xt}, 0\right)$ and radius of $R = \frac{x\sqrt{xt}}{1-xt}$ on the complex plane $z$. Again, $z = 0$ is out of the contour $C(x,t)$. Also, the variable $t$ is considered as $t < \frac{1}{x}$. Therefore, $z = x$ settles on the real axis at the inside of the circle and at the left-hand side of the center. Moreover, the poles $z = \frac{x}{1-\sqrt{xt}}$ and $z = \frac{x}{1+\sqrt{xt}}$ have been settled on the real axis as well as on the circle, at the right- and left-hand sides of center, respectively. (b) Second we suppose that $n$ is even, i.e. $n = 2k$. For a given value of $k$, once again using the integration contour $C(x,t)$, the generating functions of the second-type sequences are calculated as

$$
\begin{aligned}
G_{n=2k}(x,t) &= \sum_{m=0}^{\infty} \frac{t^m}{(2m+1)!} \frac{B^{(q,\beta)}_{m-k,\frac{q}{2}-m-k-1}(x)}{a_{m-k,\frac{q}{2}-m-k-1}(q,\beta)} \\
&= \frac{x^{\frac{q}{2}-k}}{2\sqrt{xt}} \left[ (1-\sqrt{xt})^{2k}\,\mathrm{e}^{\beta\sqrt{\frac{t}{x}}} - (1+\sqrt{xt})^{2k}\,\mathrm{e}^{-\beta\sqrt{\frac{t}{x}}} \right]. \qquad (19)
\end{aligned}
$$

Note that the series in (18) and (19) are summed on the parameter $m$ for given values $\frac{q}{2} - 2k - 2$ and $\frac{q}{2} - 2k - 1$ of $l + m$, respectively.

### 3.3. *Generating functions for given $q$ and $l - m$*

This case involves the decreasing sequences with respect to $l$ and again, there exists two possibilities: $p$ can be both odd and even. For $p = 2k + 1$ and $p = 2k$, the highest functions corresponding to them are $B^{(q,\beta)}_{k,\frac{q}{2}-k-1}(x)$ and $B^{(q,\beta)}_{k-1,\frac{q}{2}-k-1}(x)$, respectively. Also, these functions have been placed on the lines $m = -l + \frac{q}{2} - 1$ and $m = -l + \frac{q}{2} - 2$ of figure 1, respectively.

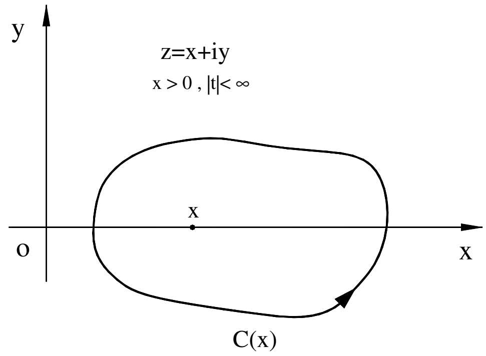


**Figure 4.** Plot of the integration contour $C(x)$ for the generating functions of third-type sequences of the associated Bessel functions.

(a) In the case that $p$ is odd, i.e. $p = 2k + 1$, for a given value of $k$ and for $|t| < \infty$, the generating functions of third-type sequences are calculated as follows:

$$\begin{aligned} G_{p=2k+1}(x,t) &= \sum_{m=0}^{\infty} \frac{t^m}{m!} \frac{B^{(q,\beta)}_{k-m,\frac{q}{2}-m-k-1}(x)}{a_{k-m,\frac{q}{2}-m-k-1}(q,\beta)} \\ &= (2k+1)!x^{\frac{q}{2}+k+1}\,\mathrm{e}^{\frac{\beta}{x}} \sum_{m=0}^{\infty} \frac{(xt)^m}{2\pi\mathrm{i}m!} \oint_{C(x)} \mathrm{d}z \frac{z^{2k-2m}\,\mathrm{e}^{\frac{-\beta}{z}}}{(z-x)^{2k+2}} \\ &= x^{\frac{q}{2}+k+1}\,\mathrm{e}^{\frac{\beta}{x}} \left[ \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k+1} \left(z^{2k}\,\mathrm{e}^{\frac{tx}{z^2}-\frac{\beta}{z}}\right)\right]_{z=x}. \end{aligned} \tag{20}$$

In order to satisfy equation (20), it is sufficient that the arbitrary contour $C(x)$ is chosen so that the points $z = x$ and $z = 0$ lay inside and outside of that, respectively (see figure 4). (b) When $p$ is an odd number, i.e. $p = 2k$, the generating functions of third-type sequences for a given $k$ are calculated as follows:

$$\begin{aligned} G_{p=2k}(x,t) &= \sum_{m=0}^{\infty} \frac{t^m}{m!} \frac{B^{(q,\beta)}_{k-m-1,\frac{q}{2}-m-k-1}(x)}{a_{k-m-1,\frac{q}{2}-m-k-1}(q,\beta)} \\ &= (2k)!x^{\frac{q}{2}+k+1}\,\mathrm{e}^{\frac{\beta}{x}} \sum_{m=0}^{\infty} \frac{(xt)^m}{2\pi\mathrm{i}m!} \oint_{C(x)} \mathrm{d}z \frac{z^{2k-2m-2}\,\mathrm{e}^{\frac{-\beta}{z}}}{(z-x)^{2k+1}} \\ &= x^{\frac{q}{2}+k+1}\,\mathrm{e}^{\frac{\beta}{x}} \left[ \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \left(z^{2k-2}\,\mathrm{e}^{\frac{tx}{z^2}-\frac{\beta}{z}}\right)\right]_{z=x}. \end{aligned} \tag{21}$$

Here, the series in (20) and (21) are summed on the parameter $m$ for given values $2k - \frac{q}{2} + 1$ and $2k - \frac{q}{2}$ of $l - m$, respectively. The accordance of the above generating functions with theorem 1 of [29] can be considered as new confirmation for it.

If we choose $q = 0$, then we can claim that relations (18)–(21) are generating functions corresponding to the associated Bessel functions with $l + m = -2(k + 1), l + m = -2k - 1, l - m = 2k + 1$ and $l - m = 2k$, respectively. Therefore, we have obtained four new different types of generating functions for the associated Bessel functions depending on whether $l + m$ and $l - m$ are negative even or negative odd integers and positive odd or non-negative even integers, respectively. Therefore, in order to obtain new generating functions we have used square-integrable associated Bessel functions in both regions $l < 0$ and $l \geqslant 0$ with the same Rodrigues representations for them.

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